PERGAMON

International Journal of Solids and Structures 38 (2001) 7543–7558

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Flexural–torsional vibration of open section composite beams with shear deformation

László P. Kollár *

Faculty of Civil Engineering, Budapest University of Technology and Economics, Bertalan L. u.2., H1521 Budapest, Hungary

Received 4 May 2000

Abstract

The paper presents the analysis of the natural frequency of thin-walled open section composite beams. Vlasov's classical theory of thin-walled beams is modified to include both the transverse shear and the restrained warping induced shear deformations. A simplified, approximate solution is also presented, in which the effect of the shear deformations are considered by Föppl's theorem. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Flexural–torsional; Vibration; Beam; Shear deformation; Torsion; Composite

1. Introduction

In a companion paper (Kollár, 2001) the governing equations of thin-walled open section composite beams were presented including the shear deformations both due to the in-plane displacements (flexural shear) and to restrained warping. A closed form solution was presented for the flexural–torsional buckling load of columns subjected to an axial force.

In this paper we apply the theory presented (Kollár, 2001) for the natural vibration of composite beams and derive closed form expressions for the period of vibration or natural frequency.

We will refer to the equations, figures, and tables of paper (Kollár, 2001) by (1-x), where x is the equation-, figure-, or table number of paper (Kollár, 2001).

2. Problem statement

We consider prismatic beams with thin-walled open cross-sections. The walls of the beams may consist of a single layer or of several layers, each layer may be made of composite materials. The layup of the walls can be unsymmetrical, however each wall must be "orthotropic", which means that axial stresses do not cause shear strains in the wall.

* Tel.: +36-1-463-1736; fax: +36-1-463-1784.
E-mail address: lkollar@eik.bme.hu (L.P. Kollár).

PII: S0020-7683(01)00025-7

The length of the beam is L, both ends are simply supported. The beam vibrates freely, the period of vibration is T, which is related to the natural frequency (f), and to the circular frequency (ω) by

$$f = \frac{1}{T} \qquad T = \frac{2\pi}{\omega} \qquad f = \frac{\omega}{2\pi}$$

We are interested in the natural frequency (period of vibration or circular frequency) of the beam.

In the analysis we assume that the beam behaves in a linearly elastic manner and the deformations are small. The contour of the cross-section of the beam does not deform in its plane; the normal stresses in the contour direction are small compared with the axial stresses. The shear deformations in the midsurface of the walls are considered.

3. Shear deformation theory

The governing equations of the shear deformation theory were presented (Kollár, 2001) and it is not reiterated here. We recall only that the shear deformations were included in the x–y and x–z planes (Timoshenko and Gere, 1961)

$$\frac{\partial v}{\partial x} = \chi_y + \gamma_y \qquad \frac{\partial w}{\partial x} = \chi_z + \gamma_z \tag{1}$$

and also in twist (Wu and Sun, 1992)

$$\frac{\partial \psi}{\partial x} = \vartheta_{\mathrm{B}} + \vartheta_{\mathrm{S}} \tag{2}$$

The first terms correspond to the case when there is no shear deformation present, and the cross-section warps, while the second terms to the case when there is only shear deformation and there is no warping.

The equilibrium equations, the strain–displacement relationships, and the constitutive equations are given by Eqs. (1-1), (1-13), and (1-16)–(1-18).

4. Flexural–torsional vibration

The equilibrium equations of a freely vibrating beam can be obtained by expressing the loads in Eq. (1-1) as follows (Weaver et al., 1990):

$$\begin{aligned} p_y &= -\rho\omega^2(v + \psi(z_{\mathrm{G}} - z_{\mathrm{sc}})) \\ p_z &= -\rho\omega^2(w - \psi(y_{\mathrm{G}} - y_{\mathrm{sc}})) \\ t &= -\rho\omega^2(v(z_{\mathrm{G}} - z_{\mathrm{sc}}) - w(y_{\mathrm{G}} - y_{\mathrm{sc}})) + \omega^2\Theta\psi \end{aligned} \tag{3}$$

where v and w are the displacements in the y and z directions, respectively, ψ is the rotation of the cross-section about the "bending deformation shear center", ω is the circular frequency, y_{sc} and z_{sc} are the coordinates of the "bending deformation shear center", ρ is the mass per unit length, y_{G} and z_{G} are the coordinates of the center of gravity of the mass from the centroid (Fig. 1), and Θ is the polar moment of mass about the "bending deformation shear center"

$$\begin{aligned} \Theta &= \rho((y_{\mathrm{G}} - y_{\mathrm{sc}})^2 + (z_{\mathrm{G}} - z_{\mathrm{sc}})^2) + \int_A m((y - y_{\mathrm{G}})^2 + (z - z_{\mathrm{G}})^2)\mathrm{d}A \\ &= \int_A m((y - y_{\mathrm{sc}})^2 + (z - z_{\mathrm{sc}})^2)\mathrm{d}A \end{aligned} \tag{4}$$

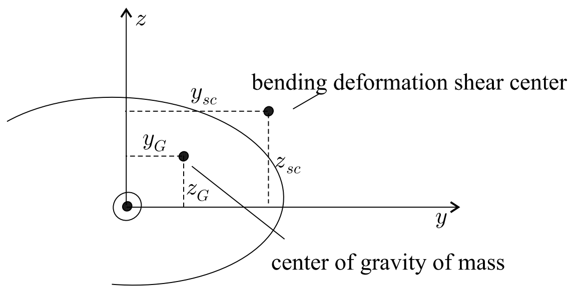


Fig. 1. Coordinates of the bending deformation shear center and the center of gravity of the mass.

where m is the mass per unit volume. To determine the circular frequency of the beam the equilibrium equations (Eqs. (1-1) and (3)), the strain–displacement relationships (Eq. (1-13)) and the constitutive equations (Eqs. (1-16)–(1-18)) must be solved taking the appropriate boundary conditions into account.

For a simply supported beam, when the rotation of the beam about the beam's axis is prevented and there are no axial constrains, the boundary conditions are (Eqs. (1-38) and (1-39))

$$v = 0 \quad w = 0 \quad \psi = 0 \quad x = 0, L \tag{5}$$

$$\frac{\partial \chi_y}{\partial x} = 0 \quad \frac{\partial \chi_z}{\partial x} = 0 \quad \frac{\partial \vartheta_B}{\partial x} = 0 \quad x = 0, L \tag{6}$$

We assume the displacements in the following form:

$$\begin{aligned} v &= v_0 \sin \alpha x \qquad & \chi_y &= \chi_{y0} \cos \alpha x \\ w &= w_0 \sin \alpha x \qquad & \chi_z &= \chi_{z0} \cos \alpha x \\ \psi &= \psi_0 \sin \alpha x \qquad & \vartheta_B &= \vartheta_{B0} \cos \alpha x \end{aligned} \tag{7}$$

where

$$\alpha = \frac{\pi}{l} \tag{8}$$

and $v_0, \ldots, \vartheta_{B0}$ are yet unknown constants. l is the half wavelength, which is

$$l = \frac{L}{k} \quad k = 1, 2, \ldots \tag{9}$$

These displacements satisfy the boundary conditions and, as will be shown below, also satisfy the differential equation system.

By introducing the displacements (Eq. (7)) into the strain–displacement relationship (Eq. (1-13)), the strains into the constitutive equations (Eqs. (1-16)–(1-18)), and the forces into the equilibrium equations (Eqs. (1-1) and (3)), we obtain from the left three equations of Eq. (1-1)

$$0 = \alpha [S_{ij}] \begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} \cos \alpha x - \left([S_{ij}] + \alpha^2 [EI_{ij}] \right) \begin{Bmatrix} \chi_{y0} \\ \chi_{z0} \\ \vartheta_{B0} \end{Bmatrix} \cos \alpha x \tag{10}$$

while from the right three equations of (Eq. (1-1))

$$0=\alpha\left[S_{ij}\right]\begin{Bmatrix}\chi_{y0}\\ \chi_{z0}\\ \vartheta_{B0}\end{Bmatrix}\sin\alpha x+\left(-\alpha^2\left[S_{ij}\right]-\alpha^2\begin{bmatrix}0&0&0\\0&0&0\\0&0&GI_t\end{bmatrix}+\omega^2\rho[M]\right)\begin{Bmatrix}v_0\\ w_0\\ \psi_0\end{Bmatrix}\sin\alpha x \tag{11}$$

In these equations $[S_{ij}]$ and $[EI_{ij}]$ are shear and bending stiffness matrices (see Eqs. (1-16) and (1-18)). $[M]$ is given by

$$[M]=\begin{bmatrix}1&0&(z_G-z_{sc})\\0&1&-(y_G-y_{sc})\\(z_G-z_{sc})&-(y_G-y_{sc})&\Theta/\rho\end{bmatrix} \tag{12}$$

Eliminating χ_{y0}, χ_{z0}, ϑ_{B0} from Eqs. (10) and (11), we obtain the following relationship:

$$0=\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&GI_t\end{bmatrix}+\left(\left[S_{ij}\right]^{-1}+\frac{1}{\alpha^2}\left[EI_{ij}\right]^{-1}\right)^{-1}-\underbrace{\omega^2}_{\lambda}\rho\frac{l^2}{\pi^2}[M]\right)\begin{Bmatrix}v_0\\ w_0\\ \psi_0\end{Bmatrix} \tag{13}$$

The non-trivial solutions of this equation gives three eigenvalues λ_i which are identical to the squares of the circular frequencies of the beam, $\lambda_i=\omega_i^2$ $(i=1,2,3)$. As a rule, all of them belong to coupled flexural–torsional vibration modes of the beam (Weaver et al., 1990).

When the cross-section has one plane of symmetry, one of the circular frequencies belongs to a flexural mode and the other two circular frequencies to flexural–torsional modes; while when the cross-section has two planes of symmetry, the three circular frequencies belong respectively to the flexural modes in the two planes of symmetry and to the pure torsional mode (when the axis of the beam does not bend).

4.1. Beams with doubly symmetrical cross-sections

We consider beams in which the cross-sections are symmetrical with respect to both the y and the z axes and the center of gravity of mass coincides with the centroid. For such beams the bending and shear deformation shear centers also coincide with the centroid and the principal directions for both bending and shear stiffnesses coincide with the y and z axes. Consequently, the bending and shear stiffness matrices simplify to

$$\left[EI_{ij}\right]=\begin{bmatrix}EI_{zz}&0&0\\0&EI_{yy}&0\\0&0&EI_\omega\end{bmatrix}\qquad\left[S_{ij}\right]=\begin{bmatrix}\widehat{S}_{yy}&0&0\\0&\widehat{S}_{zz}&0\\0&0&\widehat{S}_{\omega\omega}\end{bmatrix} \tag{14}$$

The center of gravity of mass is at the origin $(y_G-y_{sc}=z_G-z_{sc}=0)$, hence matrix $[M]$ is a diagonal matrix, and Eq. (13) simplifies to

$$0=\left(\begin{bmatrix}\frac{1}{\frac{1}{\frac{\pi^4EI_{zz}}{\rho l^4}}+\frac{1}{\frac{\pi^2\hat{S}_{yy}}{\rho l^2}}}&0&0\\0&\frac{1}{\frac{1}{\frac{\pi^4EI_{yy}}{\rho l^4}}+\frac{1}{\frac{\pi^2\hat{S}_{zz}}{\rho l^2}}}&0\\0&0&\left(\frac{1}{\frac{1}{\frac{\pi^4EI_\omega}{\Theta l^4}}+\frac{1}{\frac{\pi^2\hat{S}_{\omega\omega}}{\Theta l^2}}}+\frac{\pi^2GI_t}{\Theta l^2}\right)\frac{\Theta}{\rho}\end{bmatrix}-\omega^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{\Theta}{\rho}\end{bmatrix}\right)\begin{Bmatrix}v_0\\ w_0\\ \psi_0\end{Bmatrix} \tag{15}$$

This equation results in three circular frequencies ω_z, ω_y, and ω_ψ, which correspond to the vibration in the x–y plane, to the vibration in the x–z plane, and to the pure torsional vibration. By introducing the following definitions:

$$\omega_\psi^2 = \omega_\omega^2 + \frac{\pi^2 GI_t}{\Theta l^2} \tag{16}$$

$$\left(\omega_z^{\mathrm{B}}\right)^2 = \frac{\pi^4 EI_{zz}}{\rho l^4} \qquad \left(\omega_y^{\mathrm{B}}\right)^2 = \frac{\pi^4 EI_{yy}}{\rho l^4} \qquad \left(\omega_\omega^{\mathrm{B}}\right)^2 = \frac{\pi^4 EI_\omega}{\Theta l^4} \tag{17}$$

ω_z, ω_y, ω_ω can be calculated from Eq. (15) as follows:

$$\frac{1}{\omega_z^2} = \frac{1}{\left(\omega_z^{\mathrm{B}}\right)^2} + \frac{1}{\frac{\pi^2 \hat{S}_{yy}}{\rho l^2}} \qquad \frac{1}{\omega_y^2} = \frac{1}{\left(\omega_y^{\mathrm{B}}\right)^2} + \frac{1}{\frac{\pi^2 \hat{S}_{zz}}{\rho l^2}} \qquad \frac{1}{\omega_\omega^2} = \frac{1}{\left(\omega_\omega^{\mathrm{B}}\right)^2} + \frac{1}{\frac{\pi^2 \hat{S}_{\omega\omega}}{\Theta l^2}} \tag{18}$$

Superscript B refers to the bending deformations.

4.2. Cross-sections where the bending and shear centers coincide, and the bending and shear principal directions are identical

We take the coordinate axis in such a way that the principal directions for the bending stiffnesses lie in the y and z directions. Consequently, the bending stiffness EI_{yz} is zero $(EI_{yz} = 0)$. In addition we assume that the bending and shear centers coincide, and the bending and shear principal directions are identical. Consequently, $\widehat{S}_{yz}$, $\widehat{S}_{y\omega}$, $\widehat{S}_{z\omega}$ are zero, and the bending and the shear stiffness matrices simplify to Eq. (14). Correspondingly, Eq. (13) simplifies to

$$\begin{bmatrix} \omega^2 - \omega_z^2 & 0 & \omega^2(z_{\mathrm{G}} - z_{\mathrm{sc}}) \\ 0 & \omega^2 - \omega_y^2 & -\omega^2(y_{\mathrm{G}} - y_{\mathrm{sc}}) \\ \omega^2(z_{\mathrm{G}} - z_{\mathrm{sc}}) & -\omega^2(y_{\mathrm{G}} - y_{\mathrm{sc}}) & \left(\omega^2 - \omega_\omega^2 - \frac{\pi^2 GI_t}{\Theta l^2}\right)\frac{\Theta}{\rho} \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} = 0 \tag{19}$$

where ω_z, ω_y, ω_ω are given by Eq. (18).

When the shear deformations are neglected the shear stiffnesses have to be infinite, and Eq. (19) simplifies to

$$\begin{bmatrix} \omega^2 - \left(\omega_z^{\mathrm{B}}\right)^2 & 0 & \omega^2(z_{\mathrm{G}} - z_{\mathrm{sc}}) \\ 0 & \omega^2 - \left(\omega_y^{\mathrm{B}}\right)^2 & -\omega^2(y_{\mathrm{G}} - y_{\mathrm{sc}}) \\ \omega^2(z_{\mathrm{G}} - z_{\mathrm{sc}}) & -\omega^2(y_{\mathrm{G}} - y_{\mathrm{sc}}) & \left(\omega^2 - \left(\omega_\omega^{\mathrm{B}}\right)^2 - \frac{\pi^2 GI_t}{\Theta l^2}\right)\frac{\Theta}{\rho} \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} = 0 \tag{20}$$

which is identical to the classical solution of flexural–torsional vibration of beams.

4.3. Approximate solution

We derived a condition to calculate the natural frequencies of composite beams with arbitrary cross-sections (Eq. (13)) taking the shear deformations into account. In this section an approximate solution is presented.

We take the coordinate axis in such a way that the principal directions for the bending stiffnesses coincide with the y and z directions. Consequently, the bending stiffness EI_{yz} is zero $(EI_{yz} = 0)$. In the shear stiffness matrix we neglect the elements out of the main diagonal

$$\widehat{S}_{yz} \approx \widehat{S}_{y\omega} \approx \widehat{S}_{z\omega} \approx 0 \tag{21}$$

By this approximation Eq. (13) simplifies and the circular frequency can be calculated from Eq. (19). Note that formalistically Eq. (19) is identical to the well-known equation for calculating the circular frequencies of beams without shear deformation (Eq. (20)) if we make the following substitution

$$\begin{array}{ccc} \text{Beams without shear deformation} & & \text{Beams with shear deformation} \\ (\omega_z^{\mathrm{B}})^2 & \Rightarrow & \left(\frac{1}{(\omega_z^{\mathrm{B}})^2} + \frac{1}{\frac{\pi^2 \hat{S}_{yy}}{\rho l^2}}\right)^{-1} \\ (\omega_y^{\mathrm{B}})^2 & \Rightarrow & \left(\frac{1}{(\omega_y^{\mathrm{B}})^2} + \frac{1}{\frac{\pi^2 \hat{S}_{zz}}{\rho l^2}}\right)^{-1} \\ (\omega_\omega^{\mathrm{B}})^2 & \Rightarrow & \left(\frac{1}{(\omega_\omega^{\mathrm{B}})^2} + \frac{1}{\frac{\pi^2 \hat{S}_{\omega\omega}}{\Theta l^2}}\right)^{-1} \end{array} \tag{22}$$

ω_z^{B}, ω_y^{B}, and $\omega_\omega^{\mathrm{B}}$ correspond to vibration when the shear stiffnesses $\widehat{S}_{yy}$, $\widehat{S}_{zz}$, $\widehat{S}_{\omega\omega}$ are infinite, and GI_t is zero (Section 4.1, Eq. (17)),

- $\omega_z^{\mathrm{B}} = (\pi^4 EI_{zz}/\rho l^4)^{1/2}$ is the circular frequency if the beam vibrates in the x–y plane,
- $\omega_y^{\mathrm{B}} = (\pi^4 EI_{yy}/\rho l^4)^{1/2}$ is the circular frequency if the beam vibrates in the x–z plane, and
- $\omega_\omega^{\mathrm{B}} = (\pi^4 EI_{\omega}/\Theta l^4)^{1/2}$ is the circular frequency of the torsional vibration.

Superscript B shows that only bending deformations are considered.

It can also be shown that $\pi^2\widehat{S}_{yy}/\rho l^2$, $\pi^2\widehat{S}_{zz}/\rho l^2$, $\pi^2\widehat{S}_{\omega\omega}/\Theta l^2$ correspond to vibrations when the bending stiffnesses EI_{yy}, EI_{zz}, $EI_{\omega\omega}$ are infinite, and GI_t is zero

- $(\pi^2\widehat{S}_{yy}/\rho l^2)^{1/2}$ is the circular frequency if the beam vibrates in the x–y plane,
- $(\pi^2\widehat{S}_{zz}/\rho l^2)^{1/2}$ is the circular frequency if the beam vibrates in the x–z plane, and
- $(\pi^2\widehat{S}_{\omega\omega}/\Theta l^2)^{1/2}$ is the circular frequency of the torsional vibration.

We observe that Eq. (22) shows the same structure as the formulas suggested by Föppl (Tarnai, 1999) to determine the circular frequencies of structures characterized with different stiffnesses. It was shown that, under certain conditions, the circular frequencies of structures having two stiffnesses, D_1 and D_2, can be approximated as

$$\omega^2 \approx \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right)^{-1}$$

where ω_1 is the circular frequency of the structure if D_2 is set equal to infinity, while ω_2 is the circular frequency of the structure if D_1 is set equal to infinity.

5. Numerical examples

5.1. Buckling and vibration of an I-beam

We consider a simply supported I-beam. Its length is 800 mm, the dimensions of the cross-section are shown in Fig. (2). The width of the lower flange can take three different values: $b_2 = 60$, 90, and 120 mm.

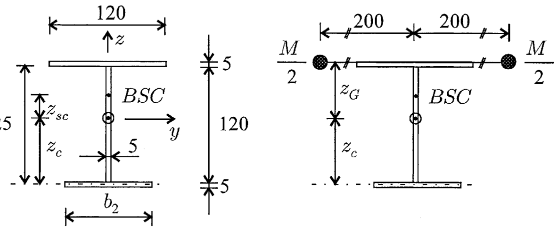


Fig. 2. Cross-section of a graphite epoxy composite I-beam.

The beam is made of unidirectional graphite epoxy, the Young modulus in the fiber direction is $E_1 = 138$ kN/mm^2 and the shear modulus is $G_{12} = 6.9$ kN/mm^2.

1. The beam is subjected to a compressive force at the end. We are interested in the intensity of the load when the beam buckles.
2. Masses are attached to the top flange of the beam as illustrated in Fig. 2. The weight of the masses are $M = (10/2)$ kg/m while the self weight of the beam is neglected. We are interested in the natural frequencies of the beam.

The properties of the cross-sections were calculated according to Table 1-1, and to the Appendix, where closed form solutions are presented for the shear stiffnesses of U, I, Z, and T beams. Note that for a single layer

$$\frac{\widetilde{\delta}_{11}}{D} = Eh \qquad \frac{\widetilde{\alpha}_{11}}{D} = \frac{Eh^3}{12} \qquad \frac{1}{\widetilde{\delta}_{66}} = \frac{Gh^3}{12} \qquad \widetilde{\alpha}_{66} = \frac{1}{Gh}$$

The beam has one axis of symmetry, consequently, we have $z_{sc} = EI_{yz} = \widehat{S}_{yz} = \widehat{S}_{z\omega} = 0$. The nonzero stiffnesses of the beam are

	$b_2 = 60$ mm	$b_2 = 90$ mm	$b_2 = 120$ mm
EI_{yy} (kN mm^2)	552×10^6	658×10^6	747×10^6
EI_{zz} (kN mm^2)	112×10^6	141×10^6	199×10^6
z_c (mm)	75.0	68.2	62.5
z_{sc} (mm)	36.1	19.7	0
z_{ssc} (mm)	−27.8	−16.5	0
i_ω^2 (mm^2)	4513	3902	3806
EI_ω (kN mm^4)	173×10^9	461×10^9	776×10^9
GI_t (kN mm^2)	0.0863×10^6	0.0949×10^6	0.104×10^6
$\widehat{S}_{yy}$ (kN)	5175	6037	6900
$\widehat{S}_{zz}$ (kN)	3850	3783	3718
$\widehat{S}_{\omega\omega}$ (kN mm^2)	$21\,962 \times 10^3$	$24\,743 \times 10^3$	$26\,953 \times 10^3$
$\widehat{S}_{y\omega}$ (kN mm)	1 43 750	99 519	0

5.1.1. Buckling load of the beam

The buckling loads of the beam were calculated both without shear deformations (Eq. (1-53)) and with shear deformations (Eq. (1-47)). The results are summarized below

	$b_2 = 60$ mm	$b_2 = 90$ mm	$b_2 = 120$ mm
$\widehat{N}^{\mathrm{B}}_{crz}$	1726	2181	3067
$\widehat{N}^{\mathrm{B}}_{cry}$	8519	10 152	11 513
$\widehat{N}^{\mathrm{B}}_{cr\omega}$	589	1821	3145
$(1/i_\omega^2)GI_t$	19.11	24.32	27.19
$\widehat{N}_{crz}$	1226	1573	2123
$\widehat{N}_{cry}$	2652	2756	2810
$\widehat{N}_{cr\omega}$	513	1392	2178
Without shear deformation (Eq. (1-53))			
$\widehat{N}_{cr1}$	2745	2965	3067
$\widehat{N}_{cr2}$	8519	10 152	11 513
$\widehat{N}_{cr3}$	538	1508	3172
With shear deformation, accurate/approximate (Eq. (1-46)/Eq. (1-52))			
$\widehat{N}_{cr1}$	1905/2018	2059/2191	2123/2123
	(1909)	(2060)	(2084)
$\widehat{N}_{cr2}$	2652/2652	2756/2756	2810/2810
	(2693)	(2817)	(2872)
$\widehat{N}_{cr3}$	485/455	1207/1130	2205/2205
	(488)	(1199)	(2190)

The approximate calculation was carried out by using Eq. (1-54) with the substitution given in expression (1-61). It can be seen that the shear deformation significantly reduces the buckling load. The buckling loads were also calculated by a finite element code (ANSYS), the results are also presented in the table in parentheses. (Note that for the given configuration the local buckling of the flanges occurs at a lower load than the global buckling.)

5.1.2. Free vibration of the beam

The mass of the beam and Θ about the bending deformation shear center are

$$\rho = 0.01 \text{ kg/mm} \qquad \Theta = \rho(200^2 + (z_{\mathrm{G}} - z_{\mathrm{sc}})^2)$$

The distance between the bending deformation shear center and the center of gravity of the mass is $z_{\mathrm{G}} - z_{\mathrm{sc}} = 127.5 - z_{\mathrm{sc}} - z_{\mathrm{c}}$ where z_{sc} and z_{c} are given in the above table. The circular frequencies of the freely vibrating beam were calculated both without shear deformation (Eq. (20)) and with shear deformation (Eq. (13)). The results are summarized below

	$b_2 = 60$ mm $\Theta = 403$	$b_2 = 90$ mm $\Theta = 416$	$b_2 = 120$ mm $\Theta = 442$
$\omega^{\mathrm{B}}_{crz}$	1.632	1.834	2.175
$\omega^{\mathrm{B}}_{cry}$	3.625	3.957	4.213
$\omega^{\mathrm{B}}_{cr\omega}$	0.319	0.513	0.646
$(\pi^2 GI_t/\Theta l^2)^{1/2}$	0.0575	0.0593	0.0601
ω_{crz}	1.375	1.557	1.810
ω_{cry}	2.022	2.062	2.082
$\omega_{cr\omega}$	0.299	0.449	0.538
Without shear deformation (Eq. (20))			
ω_{cr1}	1.638	1.873	2.297
ω_{cr2}	3.625	3.957	4.214
ω_{cr3}	0.324	0.516	0.646
With shear deformation, accurate/approximate (Eq. (13)/Eq. (19))			
ω_{cr1}	1.383/1.380	1.588/1.591	1.912/1.912
ω_{cr2}	2.022/2.022	2.062/2.062	2.082/2.082
ω_{cr3}	0.304/0.303	0.454/0.452	0.538/0.538

The approximate calculation was carried out by using Eq. (20) with the substitution given in expression (22). It can be seen that the shear deformation significantly reduces the circular frequency.

5.2. Buckling and vibration of a stiffened beam

We consider a symmetrical I-beam which is stiffened as shown in Fig. 3. The length, end conditions, material properties, and attached masses are identical to those of the previous example. We are interested in the buckling load and natural frequencies of the beam.

The results are given below

Without shear deformation		
$N_{crz} = 7483$ kN	$N_{cry} = 14196$ kN	$N_{cr\omega} = 10351$
$\omega_{crz} = 3.596$	$\omega_{cry} = 4.679$	$\omega_{cr\omega} = 1.165$
With shear deformation		
$N_{crz} = 3761$ kN (3640 kN)	$N_{cry} = 2641$ kN (2970 kN)	$N_{cr\omega} = 5188$ kN (5118 kN)
$\omega_{crz} = 2.548$	$\omega_{cry} = 2.018$	$\omega_{cr\omega} = 0.8242$

The forces in parentheses were calculated with the FE (ANSYS) code. This example further illustrates that the shear deformation due to torsion significantly reduces the buckling load and the circular frequency. (The stiffnesses of the beam are: $EI_{yy} = 920.5 \times 10^6$ kN mm^2, $EI_{zz} = 485.2 \times 10^6$ kN mm^2, $EI_{\omega} = 2548 \times 10^9$ kN mm^4, $GI_t = 0.120750 \times 10^6$ kN mm^2, $i^2_{\omega} = 3808$ mm, $\widehat{S}_{yy} = 7561$ kN, $\widehat{S}_{zz} = 3245$ kN, $\widehat{S}_{\omega\omega} = 39239$ kN.)

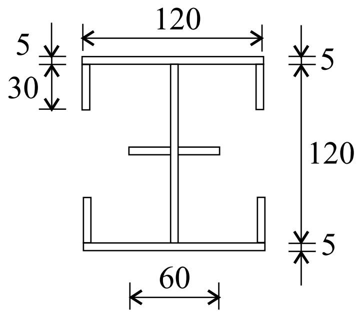


Fig. 3. Cross-section of a graphite epoxy stiffened beam.

6. Conclusion

In this paper we applied the governing equations of thin-walled open section composite beams (Kollár, 2001) including the effect of shear deformations both due to the in-plane displacements and to restrained warping for calculating the circular frequencies of freely vibrating beams. Closed form solutions were derived for the simply supported beams (Eq. (13)), and an approximate solution was also suggested, in which the well known solution of beams without shear deformations (Eq. (20)) can be used by simply reducing three terms due to the shear deformations (Eq. (22)). This solution has the advantage that it shows directly the effect of shear deformation on the circular frequency.

Acknowledgements

This work was supported by the Hungarian Science Foundation (OTKA no. T032053) which is highly appreciated.

Appendix A. Shear stiffnesses of composite beams of I, U, Z and T cross-sections

Here we determine the elements of the shear compliance matrix $(\widehat{s}_{ij})$ of four important cross-sections. We assume that the coordinates of the centroid (denoted by a dot in a circle) and the location of the shear center (=bending deformation shear center, denoted by a dot) are known (see Table 1-1).

A.1. I-beam

We consider an I-beam (Fig. 4) which has one axis of symmetry. The web is symmetrical, the top and bottom flanges can be unsymmetrical. The dimensions of the cross-section are shown in Fig. 4.

The shear flows from unit shear forces $\widehat{V}_z = 1$ and $\widehat{V}_y = 1$ acting at the (bending deformation) shear center, and from a unit torque $\widehat{T} = 1$, are given in Fig. 4. In the calculation we made the approximation that the shear flow in the web from $\widehat{V}_z$ is constant. The intensities were calculated from the equilibrium equations. δ_{sc} is calculated from the location of the (bending deformation) shear center

$$\delta_{sc} = \frac{d - (z_c + z_{sc})}{(z_c + z_{sc})} \tag{A.1}$$

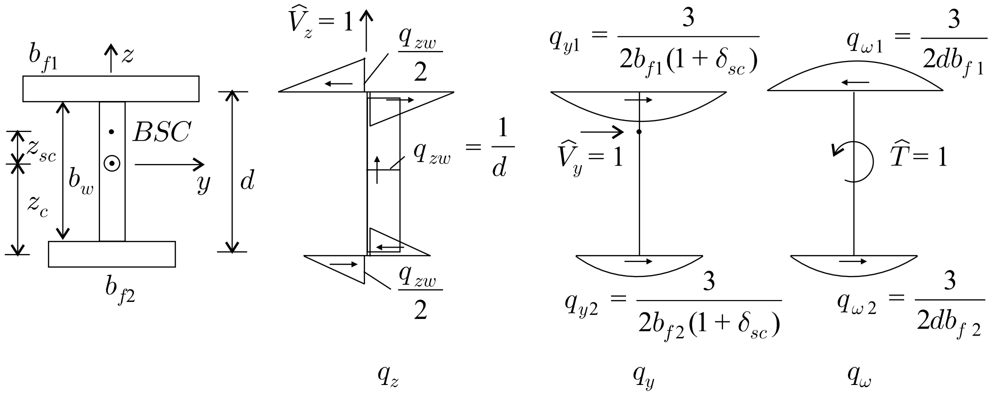


Fig. 4. I-beam and the shear flows from unit forces.

We calculate the shear compliances from (Eq. (1-27)) applying the row for q_5 of Table 1. We obtain

$$\widehat{s}_{yy} = \int \widetilde{\alpha}_{66} q_y^2 \, \mathrm{d}s = \rho \left(\frac{(\widetilde{\alpha}_{66})_{f1}}{b_{f1}(1+\delta_{\mathrm{sc}})^2} + \frac{(\widetilde{\alpha}_{66})_{f2}}{b_{f2}\left(1+\frac{1}{\delta_{\mathrm{sc}}}\right)^2} \right) \tag{A.2}$$

where $\rho = 1.2$

$$\widehat{s}_{zz} = \int \widetilde{\alpha}_{66} q_z^2 \, \mathrm{d}s = \frac{(\widetilde{\alpha}_{66})_w}{d} + \frac{1}{12}\frac{(\widetilde{\alpha}_{66})_{f1} b_{f1}}{d^2} + \frac{1}{12}\frac{(\widetilde{\alpha}_{66})_{f2} b_{f2}}{d^2} \tag{A.3}$$

$$\widehat{s}_{\omega\omega} = \int \widetilde{\alpha}_{66} q_\omega^2 \, \mathrm{d}s = \frac{\rho}{d^2}\left(\frac{(\widetilde{\alpha}_{66})_{f1}}{b_{f1}} + \frac{(\widetilde{\alpha}_{66})_{f2}}{b_{f2}} \right) \tag{A.4}$$

$$\widehat{s}_{y\omega} = \int \widetilde{\alpha}_{66} q_y q_\omega \, \mathrm{d}s = \frac{\rho}{d}\left(-\frac{(\widetilde{\alpha}_{66})_{f1}}{b_{f1}(1+\delta_{\mathrm{sc}})} + \frac{(\widetilde{\alpha}_{66})_{f2}}{b_{f2}\left(1+\frac{1}{\delta_{\mathrm{sc}}}\right)} \right) \tag{A.5}$$

$$\widehat{s}_{yz} = \int \widetilde{\alpha}_{66} q_y q_z \, \mathrm{d}s = 0 \qquad \widehat{s}_{z\omega} = \int \widetilde{\alpha}_{66} q_z q_\omega \, \mathrm{d}s = 0 \tag{A.6}$$

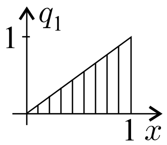
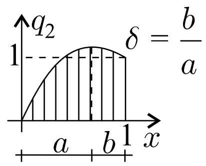
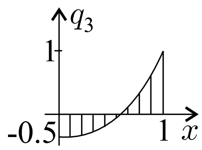
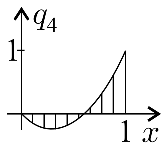
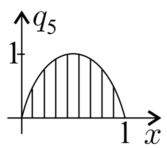
A.2. U-beam

We consider a U-beam (Fig. 5) which has one axis of symmetry. The web and the flanges may have unsymmetrical layups, but the top and bottom flanges are identical. The dimensions of the cross-section are shown in Fig. 5.

The shear flows from unit shear forces $\widehat{V}_z = 1$ and $\widehat{V}_y = 1$ acting at the (bending deformation) shear center, and from a unit torque $\widehat{T} = 1$, are given in Fig. 5. In the calculation we made the approximation that the shear flow in the web from $\widehat{V}_z$ is constant. The intensities can be calculated from the equilibrium equations. When the cross-section is subjected to $\widehat{V}_y = 1$, force equilibrium in the y direction gives

$$1 = 2\int_0^{d_f} q \, \mathrm{d}s \bigg|_{\text{Flange 1}} = 2 q_{yu} d_f \Psi_1 \tag{A.7}$$

Table 1
Integrals to evaluate the shear compliances

Function q		$\int_0^1 q\,\mathrm{d}x$	$\int_0^1 q^2\,\mathrm{d}x$
	$q_1 = x$	$\frac{1}{2}$	$\frac{1}{3}$
	$q_2 = \frac{(1+\delta)x^2-2x}{\delta-1}$	$\Psi_1 = \frac{2+3\delta-\delta^3}{3(1-\delta^2)(1+\delta)}$	$\Psi_2 = \frac{8+15\delta-10\delta^3+3\delta^5}{15(1-\delta^2)^2(1+\delta)}$
	$q_3 = \frac{3x^2-1}{2}$	0	$\frac{1}{5}$
	$q_4 = 3x^2 - 2x$	0	$\frac{2}{15}$
	$q_5 = 4x(1-x)$	$\frac{2}{3}$	$\frac{8}{15}$

$$\int_0^1 q_1 q_2\,\mathrm{d}x = \Psi_3 \qquad \int_0^1 q_1 q_3\,\mathrm{d}x = \frac{1}{8} \qquad \int_0^1 q_2 q_4\,\mathrm{d}x = \Psi_4$$

$$\Psi_3 = \frac{5+12\delta+6\delta^2-4\delta^3-3\delta^4}{12(1+\delta)^2(1-\delta^2)} \qquad \Psi_4 = \frac{2-\delta-3\delta^2}{30(1-\delta^2)}$$

$$\frac{\Psi_2}{\Psi_1^2} = \rho \approx 1.2 \qquad \frac{1}{\Psi_1} \approx \frac{3-\delta-2\delta^2}{2} \approx \frac{3(1-\delta)}{2}$$

$$\frac{\Psi_3}{\Psi_1} \approx \frac{15-\delta-2\delta^2}{24} \approx \frac{5-\delta}{8} \qquad \frac{\Psi_4}{\Psi_1} \approx \frac{3-2\delta-4\delta^2}{30}$$

$\Psi_2/\Psi_1^2 = \rho$ varies in a narrow range from 1.125 to 1.2 while the second order approximate expressions (for $1/\Psi_1$, Ψ_3/Ψ_1, and Ψ_4/Ψ_1) are practically identical to the accurate expressions when $0 \leqslant \delta \leqslant 1$.

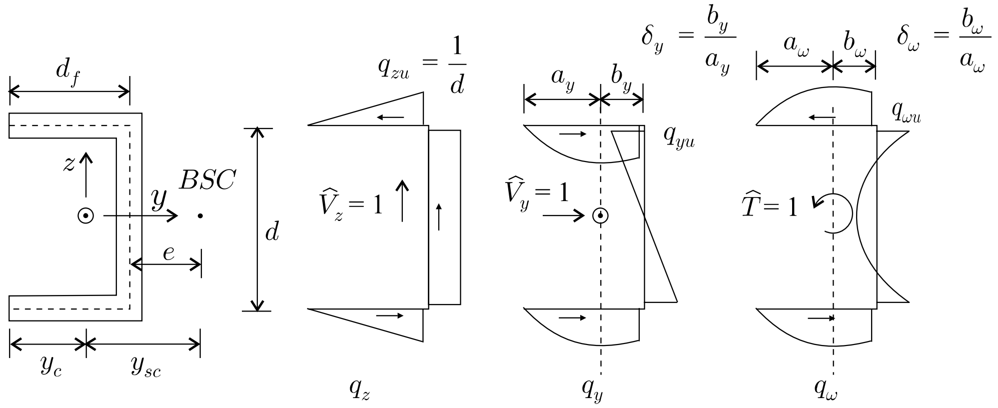


Fig. 5. U-beam and the shear flows from unit forces.

where Ψ_1 is given in Table 1 and must be evaluated at $\delta = \delta_y$. From Eq. (A.7) we obtain

$$q_{yu} = \frac{1}{2d_f \Psi_1} \approx \frac{1}{2d_f} \frac{3 - \delta_y - 2\delta_y^2}{2} \tag{A.8}$$

δ_y can be obtained from the fact that the maximum of the shear flow coincide with the location of zero stress from bending, and consequently with the centroid

$$\delta_y = \frac{d_f - y_c}{y_c} \tag{A.9}$$

When the cross-section is subjected to $\widehat{V}_z = 1$, force equilibrium in the z direction gives $q_{zc} = 1/d$. Furthermore, when the cross-section is subjected to a unit torque $\widehat{T} = 1$, the moment equilibrium yields

$$1 = d \int_0^{d_f} q \, \mathrm{d}s \Big|_{\text{Flange 1}} = d q_{\omega u} d_f \Psi_1 \tag{A.10}$$

where Ψ_1 is given in Table 1 and must be evaluated at $\delta = \delta_\omega$. From Eq. (A.10) we obtain

$$q_{\omega u} = \frac{1}{d d_f \Psi_1} \approx \frac{1}{d d_f} \frac{3 - \delta_\omega - 2\delta_\omega^2}{2} \tag{A.11}$$

δ_ω can be obtained from the fact that the maximum of the shear flow coincide with the zero axial stress from torque. The axial force for unit width is proportional to the area swept out by a generator, rotating about the center of twist, and to the axial stiffness of the wall δ_{11}/D (Table 1-1). Hence we can write (Fig. 6)

$$0 = e \frac{d}{2} \left(\frac{\widetilde{\delta}_{11}}{D} \right)_w - \frac{d}{2} b_\omega \left(\frac{\widetilde{\delta}_{11}}{D} \right)_f \tag{A.12}$$

where e is the location of the (bending deformation) shear center, as indicated in Fig. 6. We have (Fig. 6 and Eq. (A.12))

$$\delta_\omega = \frac{b_\omega / d_f}{1 - b_\omega / d_f} \qquad \frac{b_\omega}{d_f} = \frac{e}{d_f} \frac{\left(\frac{\tilde{\delta}_{11}}{D} \right)_w}{\left(\frac{\tilde{\delta}_{11}}{D} \right)_f} \tag{A.13}$$

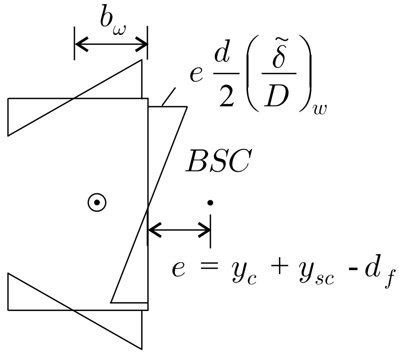


Fig. 6. Distribution of the force per unit width in a C-beam.

For symmetrical layup δ_{11}/D can be replaced by $1/a_{11}$, while for a wall consists of a single layer, δ_{11}/D can be replaced by *Eh*.

The shear compliances can be calculated as follows (Eq. (1-27))

$$\hat{s}_{yy} = \int \tilde{\alpha}_{66} q_y^2 \,\mathrm{d}s = 2\Psi_2 d_f q_{yu}^2 (\tilde{\alpha}_{66})_f + d\frac{1}{3} q_{yu}^2 (\tilde{\alpha}_{66})_w = \frac{1}{2}\underbrace{\frac{\Psi_2}{\Psi_1^2}}_{\rho\approx 1.2}\frac{(\tilde{\alpha}_{66})_f}{d_f} + d\frac{1}{3} q_{yu}^2 (\tilde{\alpha}_{66})_w \tag{A.14}$$

$$\hat{s}_{zz} = \int \tilde{\alpha}_{66} q_z^2 \,\mathrm{d}s = \frac{(\tilde{\alpha}_{66})_w}{d} + \frac{2}{3}\frac{(\tilde{\alpha}_{66})_f d_f}{d^2} \tag{A.15}$$

$$\hat{s}_{\omega\omega} = \int \tilde{\alpha}_{66} q_\omega^2 \,\mathrm{d}s = \frac{2}{d^2}\underbrace{\frac{\Psi_2}{\Psi_1^2}}_{\rho\approx 1.2}\frac{(\tilde{\alpha}_{66})_f}{d_f} + d\frac{1}{5} q_{\omega u}^2 (\tilde{\alpha}_{66})_w \tag{A.16}$$

$$\hat{s}_{z\omega} = \int \tilde{\alpha}_{66} q_z q_\omega \,\mathrm{d}s = \frac{2}{d^2}\frac{\Psi_3}{\Psi_1}(\tilde{\alpha}_{66})_f \approx \frac{2}{d^2}(\tilde{\alpha}_{66})_f \frac{15 - \delta_\omega - 2\delta_\omega^2}{24} \tag{A.17}$$

$$\hat{s}_{yz} = \int \tilde{\alpha}_{66} q_y q_z \,\mathrm{d}s = 0 \qquad \hat{s}_{y\omega} = \int \tilde{\alpha}_{66} q_z q_\omega \,\mathrm{d}s = 0 \tag{A.18}$$

A.3. Z-beam

We consider a Z-beam (Fig. 7) which is symmetrical with respect to its centroid. The layup of the web is symmetrical and the flanges may have unsymmetrical layups. The dimensions of the cross-section are shown in Fig. 7.

The shear flows from unit shear forces $\hat{V}_z = 1$ and $\hat{V}_y = 1$ acting at the (bending deformation) shear center, and from a unit torque $\hat{T} = 1$, are given in Fig. 7. In the calculation we made the approximation that the shear flow in the web from $\hat{V}_z$ is constant. The intensities can be calculated from the equilibrium equations. The detailed analysis is not given here. The results are

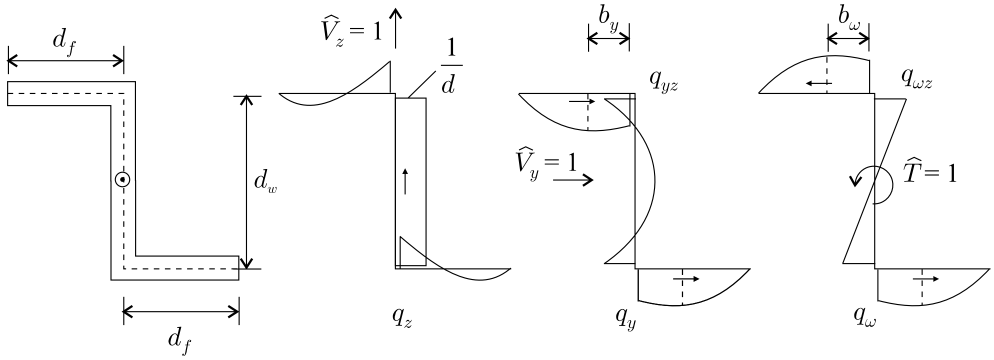


Fig. 7. Z-beam and the shear flows from unit forces.

$$q_{yz} = \frac{1}{2d_f \Psi_1} \approx \frac{1}{2d_f} \frac{3 - \delta_y - 2\delta_y^2}{2} \tag{A.19}$$

$$q_{\omega z} = \frac{1}{dd_f \Psi_1} \approx \frac{1}{dd_f} \frac{3 - \delta_\omega - 2\delta_\omega^2}{2} \tag{A.20}$$

$$\delta_y = \frac{b_y/d_f}{1 - b_y/d_f} \qquad \frac{b_y}{d_f} = \frac{1}{2 + \frac{1}{3} \frac{d\left(\frac{\tilde{\delta}_{11}}{D}\right)_w}{d_f\left(\frac{\tilde{\delta}_{11}}{D}\right)_f}} \tag{A.21}$$

$$\delta_\omega = \frac{b_\omega/d_f}{1 - b_\omega/d_f} \qquad \frac{b_\omega}{d_f} = \frac{1}{2 + \frac{d\left(\frac{\tilde{\delta}_{11}}{D}\right)_w}{d_f\left(\frac{\tilde{\delta}_{11}}{D}\right)_f}} \tag{A.22}$$

The shear compliances can be calculated as follows (Eq. (1-26))

$$\widehat{s}_{yy} = \int \widetilde{\alpha}_{66} q_y^2 \, \mathrm{d}s = \frac{1}{2} \underbrace{\frac{\Psi_2}{\Psi_1^2}}_{\rho \approx 1.2} \frac{(\widetilde{\alpha}_{66})_f}{d_f} + d\frac{1}{5} q_{yz}^2 (\widetilde{\alpha}_{66})_w \tag{A.23}$$

$$\widehat{s}_{zz} = \int \widetilde{\alpha}_{66} q_z^2 \, \mathrm{d}s = \frac{(\widetilde{\alpha}_{66})_w}{d} + \frac{4}{15} \frac{(\widetilde{\alpha}_{66})_f d_f}{d^2} \tag{A.24}$$

$$\widehat{s}_{\omega\omega} = \int \widetilde{\alpha}_{66} q_\omega^2 \, \mathrm{d}s = \frac{2}{d^2} \underbrace{\frac{\Psi_2}{\Psi_1^2}}_{\rho \approx 1.2} \frac{(\widetilde{\alpha}_{66})_f}{d_f} + d\frac{1}{3} q_{\omega z}^2 (\widetilde{\alpha}_{66})_w \tag{A.25}$$

$$\widehat{s}_{yz} = \int \widetilde{\alpha}_{66} q_y q_z \, \mathrm{d}s = \frac{\Psi_4}{\Psi_1} \frac{(\widetilde{\alpha}_{66})_f}{d} \approx \frac{(\widetilde{\alpha}_{66})_f}{d} \frac{3 - 2\delta - 4\delta^2}{30} \tag{A.26}$$

$$\widehat{s}_{y\omega} = \int \widetilde{\alpha}_{66} q_y q_\omega \, \mathrm{d}s = 0 \qquad \widehat{s}_{y\omega} = \int \widetilde{\alpha}_{66} q_y q_\omega \, \mathrm{d}s = 0 \tag{A.27}$$

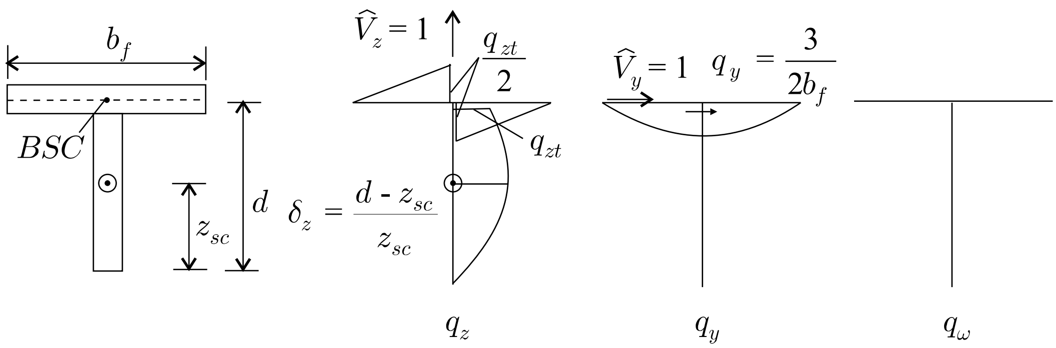


Fig. 8. T-beam and the shear flows from unit forces.

A.4. T-beam

We consider a T-beam (Fig. 8) which is symmetrical with respect to the z axis. The layup of the web is symmetrical and the flange may have unsymmetrical layup. The dimensions of the cross-section are shown in Fig. 8.

The shear flows from unit shear forces $\widehat{V}_z = 1$ and $\widehat{V}_y = 1$ acting at the (bending deformation) shear center are given in Fig. 8. The torque $\widehat{T}$ results in a distributed moment, but q_ω is zero. The intensities can be calculated from the equilibrium equations. The detailed analysis is not given here. The results are

$$\widehat{s}_{zz} = \int \widetilde{\alpha}_{66} q_z^2 \, \mathrm{d}s = \frac{(\widetilde{\alpha}_{66})_w}{b_w} \underbrace{\frac{\Psi_2}{\Psi_1^2}}_{\rho \approx 1.2} + \frac{1}{12} (\widetilde{\alpha}_{66})_f b_f q_{zt}^2 \tag{A.28}$$

$$\widehat{s}_{yy} = \int \widetilde{\alpha}_{66} q_z^2 \, \mathrm{d}s = \frac{(\widetilde{\alpha}_{66})_f}{b_f} \rho \tag{A.29}$$

$$\widehat{s}_{\omega\omega} = \infty \tag{A.30}$$

$$\widehat{s}_{zy} = \widehat{s}_{\omega y} = \widehat{s}_{\omega z} = 0 \tag{A.31}$$

q_{zt} is the shear flow at the top of the web, defined as

$$q_{zt} = \frac{1}{d\Psi_1} \approx \frac{3 - \delta_z - 2\delta_z^2}{2d} \tag{A.32}$$

δ_z is a parameter specifying the position of the centroid (Fig. 8).

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